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Dispersion of short pulses: Guigay matrix

J. Ojeda-Castaneda,*[1] and Cristina M. Gómez-Sarabia [2]

[1] *Electronics Department, Campus Salamanca, University of Guanajuato, 36885 Guanajuato, México*
[2] *Digital Arts, Campus Salamanca, University of Guanajuato, 36885 Guanajuato, México*

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Abstract—By employing Guigay coefficients, one can describe in an elegant and useful manner the Fresnel diffraction patterns of a periodic structure. Here, we relate Guigay formulation with the classical Fourier series treatment of first order dispersion. Then, we propose the use of a remarkably simple matrix for describing first order dispersion.

Fresnel diffraction patterns of periodic structures find usefully applications when setting Talbot interferometers [1, 2], for implementing array illuminators [3, 4], for designing theta decoders [5, 6], for measuring the degree of spatial coherence [7, 8], and for evaluating Fresnel similarity [9].

By taking advantage of the analogy between Fresnel diffraction and first order dispersion [10], one can describe temporal intensities [11] and temporal similarities [12], at fractional distances of the Talbot length.

Several years ago, Guigay identified certain propagation coefficients (here denoted as Guigay coefficients) for describing in an elegant manner the Fresnel diffraction patterns of a grating [13].

Table 1. Our present scope.

Fractional Talbot effect	*Fourier Series*	*Matrix Description (Guigay coefficients)*
Space	*Well-known*	*Reference 16*
Time	*Well-known*	*This paper*

As indicated in Table 1, here our aim is twofold. First, we relate Guigay formulation with the classical Fourier series treatment of Fresnel diffraction. Second, since to our knowledge there is not a matrix description of first order dispersion, then we suggest using a remarkably simple matrix treatment of this physical phenomenon.

* E-mail: jojedacas@ugto.mx

As is depicted in Fig. 1, at $z = 0$, we assume that a plane wavefront illuminates a grating, with period d, whose complex amplitude transmittance is

$$g(x) = \sum_{m=-\infty}^{\infty} c_m \exp\left(i2\pi x \frac{m}{d} \right). \tag{1}$$

From Eq.(1), it is straightforward to obtain the complex amplitude distribution of the Fresnel diffraction patterns at a fractional Talbot distance, $z = (2\,d^2/\lambda)\,/\,N = Z_T\,/\,N$,

$$\mathrm{u}(x, z = \frac{2d^2}{\lambda N}) = \sum_{m=-\infty}^{\infty} c_m \exp\left(-i2\pi \frac{m^2}{N} \right) \exp\left(i2\pi x \frac{m}{d} \right). \tag{2}$$

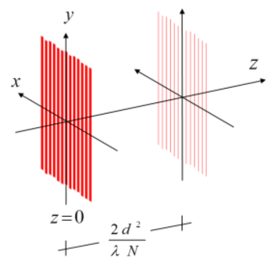

Fig. 1. Fractional Talbot effect.

In Ref. [13], Guigay has proved that the complex amplitude distribution in equation 2 can also be written as

$$\mathrm{u}(x, \frac{2d^2}{\lambda N}) = \sum_{n=0}^{N-1} D_n(\mathrm{N})\, g(x - \frac{n}{N}\mathrm{d}). \tag{3}$$

The coefficients in Eq. (3) are here denoted as the Guigay coefficient, which are to be specified in what follows. If we substitute Eq. (1) into Eq. (3), we obtain

$$\mathrm{u}(x, \frac{2d^2}{\lambda N}) = \sum_{m=-\infty}^{\infty} c_m \left\{ \sum_{n=0}^{N-1} \mathrm{D}_n(\mathrm{N}) \exp\left(-i2\pi \frac{m\,n}{N} \right) \right\} \exp\left(i2\pi x \frac{m}{d} \right). \tag{4}$$

By simple comparison of Eqs. (2) and (4), we conclude that

$$\exp(-\mathrm{i}\,2\pi \frac{m^2}{N}) = \sum_{n=0}^{N-1} \mathrm{D}_n(\mathrm{N}) \exp\left(-i2\pi \frac{m\,n}{N}\right). \quad (5)$$

Equation (5) is to be recognized as a discrete Fourier transformation. Hence, the Guigay coefficients can be obtained by taking the discrete inverse Fourier transform of Eq. (5) to give

$$\mathrm{D}_n(\mathrm{N}) = \frac{1}{N} \sum_{n=0}^{N-1} \exp(-\mathrm{i}\,2\pi \frac{m^2}{N}) \exp\left(i2\pi \frac{m\,n}{N}\right). \quad (6)$$

The result in equation 6 agrees well with the treatment of self-images given by Winthrop and Worthington [14].

The remarkable feature of Guigay contribution is that the coefficient have the following explicit, analytical expression

$$\mathrm{D}_n(\mathrm{N}) = \frac{\exp(-i\frac{\pi}{4})}{\sqrt{2N}} \left[1 + (-1)^n (i)^N\right] \exp(i\frac{\pi n^2}{2N}). \quad (7)$$

From Eq. (7), we recognize that the coefficients are periodic, with period N. Furthermore, from the Argand diagram in Fig. 2, it is apparent that for a given value of *n* (say *n*=0) the coefficients can have only 4 different values.

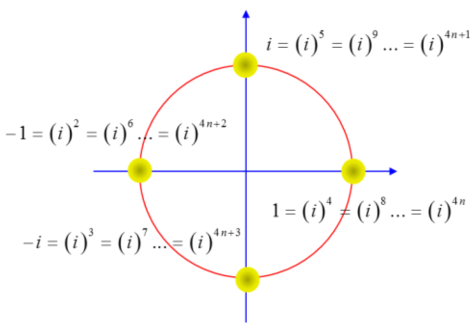

Fig. 2. Argand diagram of the complex number $(i)^N$.

The above property is a feature of the Gauss sum [15], which is easily evaluated by setting $n = 0$ in Eq. (7).

In Fig. 3, we use the Argand diagram for plotting the values of the Gauss sum, which are obtained by evaluating either equation 6 (Winthrop and Worthington) or Eq. (7) (Guigay coefficients) if $N \leq 100$.

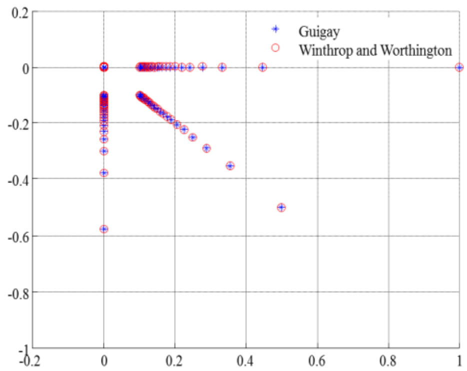

Fig. 3. Argand diagram of Guigay coefficient $D_0(N)$, for $N \leq 100$.

Let us consider that at the input, $z = 0$, of a dispersive medium the input complex amplitude envelope is represented by a periodic function. At the distance *z*, inside a first-order dispersion medium (with dispersion coefficient β_2), the slowly varying complex amplitude envelope is described by the mathematical expression

$$\psi(\tau, z) = \sum_{m=-\infty}^{\infty} C_m \exp\left[-i\,2\pi z (\Omega^2 \beta_2 / 4\pi) m^2 + i2\pi\, m\Omega\tau\right]. \quad (8)$$

In Eq. (8) the Greek letter $\Omega = 1/T$ is the temporal frequency of the slowly varying complex amplitude envelope. The Greek letter τ denotes the time measured in the proper reference frame. At the fractional Talbot distance $z = Z_T / N = [4\pi / (\Omega^2 \beta_2)] / N$, Eq. (8) becomes

$$\psi(\tau, \frac{4\pi}{\Omega^2 \beta_2 N}) = \sum_{m=-\infty}^{\infty} C_m \exp\left[-i\,2\pi \frac{m^2}{N} + i2\pi\, m\Omega\tau\right]. \quad (9)$$

By a simple comparison between Eqs. (2) and (9), and taking into account the result in Eq. (3), we recognize that Eq. (9) can also be written as

$$\psi(\tau, \frac{1}{\pi \Omega^2 \beta_2 N}) = \sum_{n=0}^{N-1} D_n(\mathrm{N})\, \psi_0(\tau - \frac{n}{N\Omega}). \quad (10)$$

In other words, at the fractional Talbot distance $z = Z_T / N$, the slowly varying complex amplitude envelope can be expressed as a linear combination of laterally shifted versions of the initial complex amplitude envelope. This result is no so well-known. In what follows we propose a matrix treatment, which is similar to those discussed elsewhere for describing either Fresnel diffraction [16], or image formation with noncoherent illumination [17].

For the sake of simplicity in our present treatment, we discuss an illustrative example. However, by mathematical induction, the generic case follows the same procedure. In Fig. 4 we show pictorially the result in Eq. (10), for the case N=5.

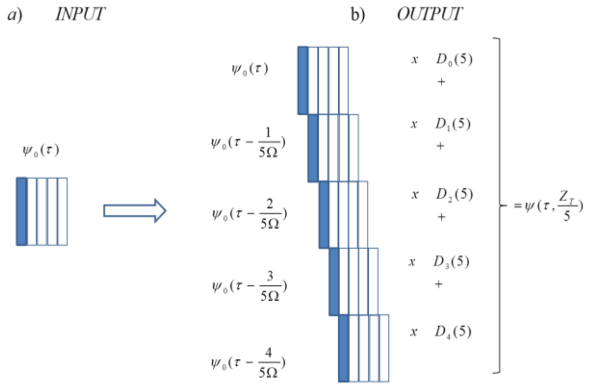

Fig. 4. Pictorial representation of Eq. (10) for N = 5.

The input complex amplitude envelope is represented by a rectangular window, which fills only one fifth of the period T/N=1/(5Ω). Since the initial pulse is fills only 1/5 of the total width, we can relate the temporal delays with the row vectors [1, 0, 0, 0, 0], [0, 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 1, 0], and [0, 0, 0, 0, 1]; which form an orthonormal base.

$$\psi\left(\tau, \frac{Z_T}{5}\right) = \begin{pmatrix} D_0(5) & D_4(5) & D_3(5) & D_2(5) & D_1(5) \\ D_1(5) & D_0(5) & D_4(5) & D_3(5) & D_2(5) \\ D_2(5) & D_1(5) & D_0(5) & D_4(5) & D_3(5) \\ D_3(5) & D_2(5) & D_1(5) & D_0(5) & D_4(5) \\ D_4(5) & D_3(5) & D_2(5) & D_1(5) & D_0(5) \end{pmatrix} \psi_0(\tau) \qquad (11)$$

Equation (11) defines the Guigay matrix, which contains as matrix elements the Guigay coefficients. Due to the cyclic structure of the matrix, one can extend the result in Eq. (11), to other values of N. Of course, we can express Eq. (11) as an operator, which is denoted as the evolution operator

$$\psi(\tau, \frac{Z_T}{N}) = D(N)\ \psi_0(\tau). \qquad (12)$$

It is straightforward, yet outside our present scope, to show that

$$D^{-1}(N) = \left[D^t(N)\right]^*; \quad D(N) = D^t(N); \quad D^{-1}(N) = D^*(N). \qquad (13)$$

See Ref. [16, 17]. In Eq. (13) the super index t denotes the transposing operation. Since the whole process is reversible, one can use the operator in Eq. (12) for identifying phase modulations, which are identical to the output complex amplitude envelope. Then, by inverse evolution (Eq. (13)) these phase modulations can be used to generate short pulses filling only 1/N of the total period.

In conclusion, we have related the Guigay formulation with the classical Fourier series treatment. Then, we have applied Guigay coefficients for obtaining a remarkably simple matrix treatment of first order dispersion; which is of course related to the matrix treatment of Fresnel diffraction. The matrix describes the evolution operator of first order dispersion.

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