Solution to the Boundary problem for Fourier and Multigrid transport of intensity equation based solvers

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Abstract—In this work we present a solution for the boundary problem for phase retrieval techniques based on the Transport of Intensity Equation (TIE). The solution presented here is based on the Neumann Boundary condition and the mirror padding scheme of the captured intensities. The obtained results are derived for the widely used Fourier Transform based TIE solver, but it is shown that they can also be applied to Multigrid based techniques.

Quantitative phase imaging (QPI) has been used extensively for visualizing hidden features of biological and technical samples [1–3]. For this reason, phase retrieval techniques have been employed in science and technology with a high rate of success and are still a vivid research field. Phase retrieval techniques based on interferometry allow quantitative phase reconstructions [2]. However, optical setups used for these techniques require coherent illumination, interferometer stability, and the retrieved phase needs to be unwrapped. Single beam phase retrieval techniques seek to obtain the phase of a wave-field from a sequence of through focus intensities [1, 4]. Within this family, phase retrieval techniques based on the Transport of Intensity Equation (TIE) [1] have gained increased interest because they relate the transversal energy flux of the phase with the axial energy flux of intensity in the Fresnel region by means of a linear transformation [5]. Moreover, TIE based algorithms give a unique solution plus an arbitrary constant, the retrieved phase does not need to be unwrapped and can be employed to partially coherent illumination [5]. However, a major disadvantage of TIE based solvers is that these algorithms are strongly affected by Low Frequency Artifacts (LFAs). A common source of LFAs in TIE methods are caused by the intrinsic property of amplifying low frequency noise. One solution to deal with this problem is to choose the proper plane separation strategy for capturing the intensities [6–8]. Reference [7] shows that for equally spaced planes there is an optimal distance that reduces the impact of LFAs. However, in order to have efficient minimization of LFA, a large number of planes have to be captured. In a recent publication [8], it was shown that the exponential separation strategy [8] can mitigate the LFAs on TIE methods by using only a few captured intensities [9].

Most of the literature in TIE is dedicated to improve the accuracy of TIE algorithms by optimizing the sampling in the axial direction. Often it is neglected that TIE is a second order partial differential equation and boundary conditions (BC) are required to solve this equation. When the phase distribution does not interact with the edges of the field of view (FoV), the selection of the BC is not important [10]. However, for phase distributions that interact with the edges of the FoV, the selection of the BC becomes crucial to solve TIE properly. A wrong selection of the BC will result in strong LFAs across the retrieved phase that cannot be suppressed with the capturing of additional images. In reference [10], it is shown the selection rules for the BC for extended phase object. However, this solution requires strict symmetrical properties and cannot be applied to more general cases. In reference [11], it is shown that the employment of an aperture at the object plane and the Neumann BC will give a proper solution for TIE. The work in reference [12] takes into account the previous considerations and implements them into a TIE solver based on the Discrete Cosine Transform (DCT). Reference [12] shows that the advantage of this approach lies in the fact that the NBC can be directly integrated into the DCT-TIE. In this work, we extend the applicability of the aperture and the NBC to the TIE solvers based on the Fourier (FT) and Multigrid (MG) approaches. We show that these methods can give an accurate solution for the Boundary problem as well. Further, we will carry out the corresponding simulations in order to prove that we extend the solution to the Boundary problem for the Fourier (FT-TIE) and the Multigrid (MG-TIE) TIE based solvers

Derived from the Helmholtz Equation when considering the paraxial approximation, TIE has the analytical form [5]:

$$\nabla \cdot \left[I_0 \nabla \varphi \right] = -k \partial_z I, \tag{1}$$

where I_0 is the on-focus intensity, φ is the phase and $\partial_z I$ is the intensity axial derivative that can be estimated using different methods [7, 9]. Equation (1) can be solved as a

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Poisson Equation limited to a rectangular domain B [5, 7, 12]. The BC can be defined as

$$\varphi(\Gamma) = f(\Gamma), \quad (2)$$

$$I_0 \frac{\partial}{\partial \mathbf{n}} \varphi(\Gamma) = g(\Gamma), \quad (3)$$

where Γ are the boundary values of the domain B, *f* and *g*, are arbitrary functions, and **n** is the normal vector in the outward direction of the FoV. Equation (2) is Dirichlet BC (DBC), and (3) is the Neumann BC (NBC). For the Periodic BC (PBC), it is assumed that the values at the borders of the detector are nearly similar, which implies a periodic flux of energy. A partial solution to deal with the BC problem consists in putting a complex object in the center of the FoV in order to have soft-edged intensity distribution across the boundary of the detector. When using this approach, the functions *f* and *g* in Equations (2) and (3) can be considered close to zero, and thus, any BC can be selected without any loss of generality. Another partial solution for the Boundary problem consists in observing the symmetrical properties $\partial_z I$, and thus, selecting the convenient BC [10]. These two solutions rely on the fact that there is no leak of energy through the boundaries of the detector. However, these solutions cannot be applied to a more general case of extended phase objects that interact with the border of the detector.

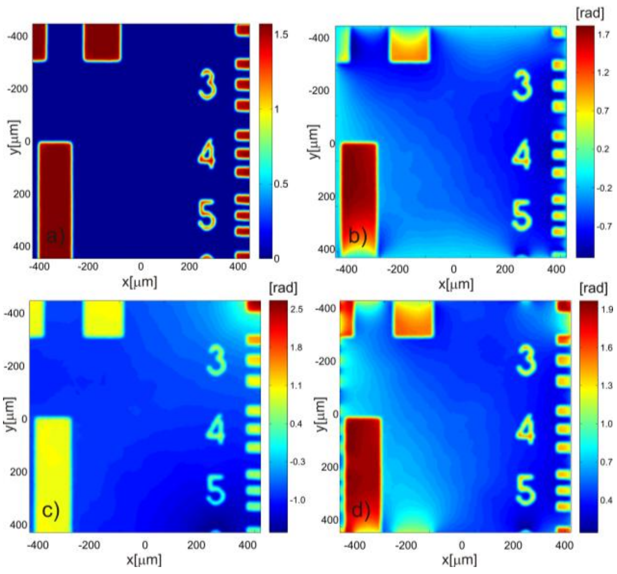

Fig. 1. LFA in the retrieved phase due to the improper selection of the BC. a) Original phase distribution. Retrieved phases when employing b) DBC, c) NBC and d) PBC.

Simulations were performed to test the FT-TIE solver for an extended phase object that touches the edge of the detector. To simulate propagated intensities, we employed the Angular Spectrum Method proposed in [13]. For these simulations eleven intensities with separations z=0, ±20, ±40, ±60, ±80, ±100μm were captured, the pixel size is Δx=3.45μm, the number of pixels in the detector is 256×256, and the phase difference is $\Delta\varphi$=0.5π. Further, the intensities are subject to Additive Uncorrelated White Gaussian Noise (AUWGN) with a Signal to Noise Ratio (SNR) of 50dB. Figure 1a shows the original phase distribution. The retrieved phases for the FT-TIE method, when employing the DBC, NBC and PBC, are shown in Figs. 1b, 1c and 1d with a Root Square Mean Error (RMSE) of 0.5rad, 0.8rad and 0.45rad, respectively.

In order to find a solution for phase objects as the one presented in Fig. 1a, let P be a square aperture at the object plane defined as

$$P = \begin{cases} 1 & \vec{r} \in A, \\ 0 & \vec{r} \notin A \end{cases} \quad (4)$$

where $\boldsymbol{r}=(x,y)$ is the transversal coordinate vector and A is a square domain smaller than B. Thus, Eq. (1) can be re-expressed as

$$I\nabla^2\varphi + \nabla I \cdot \nabla\varphi + I_0 \hat{n} \cdot \nabla\varphi = -k\partial_z I, \quad (5)$$

where *I* is the intensity inside of the aperture. Comparing the left hand side of (3) and the third term of (5), we realize that these are the same. The purpose of this aperture is to enforce TIE to follow the Law of conservation of energy when the phase distribution is interacting with the borders of the FoV. For this case, it has been shown that only the NBC will give a proper solution to Eq. (5) [11, 12]. In order to solve (5), reference [12] employs a TIE solver based on the DCT arguing that FT approach is unable to give a proper solution when using NBC. This claim is supported by the fact that the FT based methods possess an intrinsic Periodic BC, and thus, only by putting the aperture on the on-focus plane will not modify the boundary property of the FT-TIE solver. However, it is shown in reference [14] that the FT-TIE solver can be provided with the DBC and NBC through appropriate mirror padding of captured intensities. Figure 2a shows the in-focus intensity distribution I_0 with an aperture. The black frame in Fig. 2a indicates that the intensity is zero for those pixels. For solving (5) with the FT-TIE, it is necessary to place the aperture on the object plane and capture the corresponding through focus intensities. Fig. 2b shows the DBC mirror padding scheme [14] for the captured intensities. The right-top and left-bottom intensities in Fig. 2b are mirrored and multiplied by minus one. In order to impose the required NBC in the FT-TIE solver, it is necessary to apply the mirror padding shown in Fig 2b to captured intensities, and thus, these images will be used to estimate the axial intensity derivative [7, 9]. Later on, the padded I_0 and $\partial_z I$ will be used as inputs of the FT-TIE method [14].

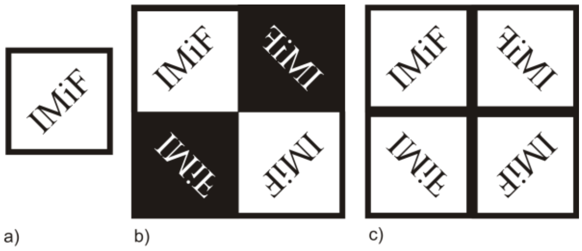

Fig. 2. Mirror padding scheme [14] for imposing BC in the FT-TIE solver. a) Intensity distribution in the object plane with an aperture. b) DBC mirror padding. c) NBC mirror padding. The black frame in Fig. 2a) indicates that the intensity is zero.

So far, we have examined the solution of Eq. (5) for the case of the FT-TIE method. However, Eq. (5) can be solved with the Multigrid approach as well. The MG-TIE, unlike the FT-TIE, retrieves the phase in the spatial domain by modeling Eq. (1) [or (5)] as a system of linear equations [10]. The MG approach employs iterative solvers and a series of down-sampled computational grids for improving the convergence of the solution [15]. An additional advantage of this approach is that the BC can be imposed explicitly inside of the solver without applying any mirror padding scheme [14] on the intensities.

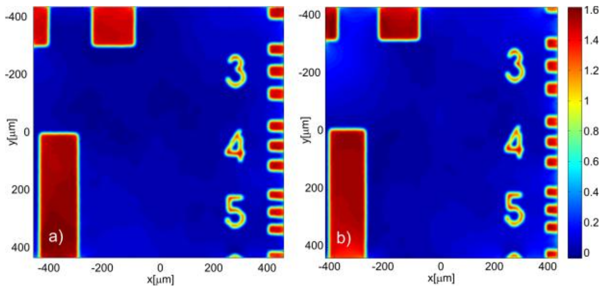

Fig. 3. Retrieved phases with the NBC. a) FT-TIE. b) MG-TIE.

In order to investigate the accuracy and effectiveness of FT-TIE and MG-TIE methods, the following simulations were carried out when employing the aperture necessary for fulfill (4) and the NBC, using the same experimental conditions as in Fig. 1. The first method to be tested is the FT approach. Figure 3a shows the retrieved phase when employing this technique. Later on, we tested the MG approach. The MG algorithm can be solved employing several techniques for improving the accuracy of the solution [16]. In this contribution, the Full MG algorithm (FMG) is used. For this solver, the number of relaxations and cycles used to compute the FMG-TIE were 80. Figure 3b shows the retrieved phase when employing the MG-TIE method. Comparing both retrieved phases (Fig. 3a and b) with the original phase (Fig. 1a) we found that the Root Square Mean Error (RMSE) is 0.03rad and 0.1rad for the FT-TIE and MG-TIE, respectively. This means that the error in the retrieved phases with an aperture decrease is at least 4 times lower with respect to the phases obtained without an aperture. The retrieved phase distributions showed in Fig. 3 prove that the employment of an aperture and the NBC removes efficiently the LFAs. In Figure 3, it can be observed that some LFAs remained; however, these LFAs are caused by noise amplification and not by the BC. Finally, a quantitative comparison between the RMSE of Figs. 3a and 3b shows that the FT approach is three times more accurate.

In this contribution we have presented that LFAs in the retrieved phase due to the BC when employing FT-TIE or MG-TIE technique can be overcome when using a physical aperture and the NBC. In the case of the FT-TIE method, we show that the captured intensities with an aperture have to be mirror padded before processing them with the FT-TIE algorithm. In the case of the MG-TIE, we showed that such symmetrization is not needed and the captured intensities can be directly processed, minimizing memory requirements. With these modifications, both techniques have shown that the retrieved phase is accurate and free from boundary artifacts.

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